University “Al. I. Cuza” Iaşi

Faculty of Mathematics

Resume of Doctoral Thesis

Optimal Control of Navier-Stokes Fluid Flows

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Iaşi
10 July, 2012

UNIVERSITY “ALEXANDRU IOAN CUZA” IN IAŞI
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On 10 July, hour....., in the conference room of Faculty of Mathematics, **Hanbing Liu** will give the public defense of his doctoral thesis titled *Optimal control of Navier-Stokes fluid flows*, in order to obtain the doctor degree of Mathematics.

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President:
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I kindly invite you to participate the public defense of the doctoral thesis.

Preface

Application of optimal control theory to flow control problems has recently attracted increased attention since the prominent works of F.Abergel and R.Temman (1990,(1)), J.A.Burns and S.Kang (1991,(2)) , M.D.Gunzburger et al. (1992,(3)), S.S.Joshi et al. (1997,(4)), T.R.Bewley and S.Liu (1998, (5)), V.Barbu and S.S.Sritharan (1998,(6)), and so on. A key element of an optimal flow control problem is the minimization of an objective or cost functional which provides a quantitative measure of the desired objective and depends critically on the solution (known as the optimal solution) that satisfies the partial differential equations governing the fluid flow. For instance, the integral of the dissipation function may be employed as an objective functional, the governing PDEs are the Navier-Stokes equations, and their optimal solution represents the flow with minimum drag on a body (e.g., (1)).

In this work, the optimal control problems of fluid flows with state constraint are investigated.

In Chapter 1, we shall study the optimal control problems of 3-D Navier-Stokes equations with state constraint of pointwise type. Strong results in 2-D are also given. The idea applied is essentially due to V.Barbu (7). Therein the maximum principle of optimal control problems with state constraint of pointwise type for linear evolution equations in Banach spaces are obtained. Since here the governing system is nonlinear equations, the arguments will be more precise and constructive. The results presented in Chapter 1 have bee published in (8).

In Chapter 2, we shall give the similar results as in Chapter 1 for optimal control problems governed by Magnetohydrodynamic equations, which describe the motion of the conductive flows in a magnetic field. Since this system is a coupling of Navier-Stokes equations and Maxwell equation, the approach to obtain the optimality system is not straight forward from the results for Navier-Stokes equations. Moreover, the optimal control problem for Magnetohydrodynamic equations contains itself great physical interesting. The results appeared in this chapter have been published in (9).

In Chapter 3, we shall study the optimal control problems governed by linearized Navier-Stokes equations, or to say, Stokes-Oseen flows. The Dirichlet boundary control and the state constrained of periodic type are considered. This unboundedness of the control problem will cause difficulty mathematically, but the control on boundary usually invokes great interest since it is sometimes more conveniently implementable in engineering. The objective of cost functional is to minimize the vorticity of the flow field, which invokes great interest physically which will be mentioned in Chapter 2 and Chapter 3. However, the way to approach the optimality system in this case will be more constructive. This is also one main difference between this work and works mentioned in Chapter 3. The results presented in Chapter 3 have been published online in (10).

In Chapter 4, the feedback form of the Dirichlet boundary optimal control for the time periodic control problems of Stokes-Oseen equations will be given. Since here the optimal control problem considered is unbounded, a prior regularity results for the optimal solution will be given. Thanks to the maximum principle and regularity results for the optimal solution, we can obtain the feedback form of the optimal control, and we can apply it to the periodic Navier-Stokes equations to refine some defect property of the periodic solutions.

In the period of doctoral study, the author is also identified as a early stagy researcher with funds support in the project
Marie Curie Initial Training Network (ITN) Call: FP7-PEOPLE-2007-1-1-ITN, no. 213841-2.
which is directed by professor Dr. Aurel Rascanu.

1

Optimal control of Navier-Stokes equations with state constraint

1.1 Introduction

In this chapter, we shall study the optimal control problem

$$(\mathbf{P})\qquad \textbf{Minimize}\ \frac{1}{2}\int_0^T\left(\int_\Omega |\mathscr{C}(\mathbf{y}(t,\mathbf{x})-\mathbf{y}^0(t,\mathbf{x}))|^2\right)d\mathbf{x}dt+\int_0^T h(\mathbf{u}(t))dt;$$

subject to

$$\begin{cases} \frac{\partial \mathbf{y}}{\partial t}-\nu\triangle\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{y}+\nabla p=D_0\mathbf{u}+\mathbf{f}_0 & \text{in } \Omega\times(0,T),\\ \mathbf{y}(0)=\mathbf{y}_0 & \text{in } \Omega,\\ \nabla\cdot\mathbf{y}=0 & \text{in } \Omega\times(0,T),\\ \mathbf{y}=0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1.1}$$

$$\mathbf{y}(t)\in K,\qquad \forall t\in(0,T), \tag{1.1.2}$$

Here Ω is a bounded and open subset of $\mathbb{R}^N(N=2,3)$ with smooth boundary $\partial\Omega$. The source field $\mathbf{f}_0\in L^2(0,T;(L^2(\Omega))^N)$. The operator $D_0\in L(U;(L^2(\Omega))^N)$, and the control function $u\in L^2(0,T;U)$, where U is a Hilbert space. The function $h:U\to(-\infty,+\infty]$ is convex and lower semicontinuous, $\mathbf{y}^0\in L^2(0,T;H)$, and the operator $\mathscr{C}\in L(V,H)$. Here K is a closed convex subset in H. where

$$H=\{\mathbf{y};\mathbf{y}\in(L^2(\Omega))^N,\nabla\cdot\mathbf{y}=0,\mathbf{y}\cdot\mathbf{n}=0\text{ on }\partial\Omega\}, \tag{1.1.3}$$

$$V=\{\mathbf{y};\mathbf{y}\in(H_0^1(\Omega))^N,\nabla\cdot\mathbf{y}=0\ \}. \tag{1.1.4}$$

We endow the space H with the norm of $(L^2(\Omega))^N$, and denote by $\langle\cdot,\cdot\rangle$ the scalar product of H, $\langle\cdot,\cdot\rangle_{(V,V')}$ the paring between V and its dual V' with the norm $\|\cdot\|_{V'}$. Denote by the symbol $\|\cdot\|$ the norm of the space V, which is defined by

$$\|\mathbf{y}\|^2=\sum_{i=1}^N\int_\Omega|\nabla y_i|^2dx,$$

and by the symbol $|\cdot|$ the norm of $\mathbb{R}^N$ and $(L^2(\Omega))^N$.

Let $P : (L^2(\Omega))^N \to H$ be the orthogonal projection on H(the Leray projector), and set

$$a(\mathbf{y}, \mathbf{z}) = \int_\Omega \nabla \mathbf{y} \cdot \nabla \mathbf{z}, \forall \mathbf{y}, \mathbf{z} \in V, \tag{1.1.5}$$

$$A = -P\triangle, D(A) = V \cap (H^2(\Omega))^N. \tag{1.1.6}$$

Equivalently,

$$\langle A\mathbf{y}, \mathbf{z}\rangle = a(\mathbf{y}, \mathbf{z}),\ \forall \mathbf{y}, \mathbf{z} \in V.$$

The stokes operator A is self-adjoint in H, $A \in L(V, V')$ and $\langle A\mathbf{y}, \mathbf{y}\rangle = \|\mathbf{y}\|^2, \forall \mathbf{y} \in V$. Finally, consider the trilinear function

$$b(\mathbf{y}, \mathbf{z}, \mathbf{w}) = \sum_{i=1}^{N} \int_\Omega y_i D_i z_j w_j dx, \tag{1.1.7}$$

and we denote by $B : V \to V'$ by

$$B\mathbf{y} = P(\mathbf{y} \cdot \nabla)\mathbf{y},\ \forall \mathbf{y} \in V, \tag{1.1.8}$$

or equivalently, $\langle B(\mathbf{y}), \mathbf{w}\rangle = b(\mathbf{y}, \mathbf{y}, \mathbf{w}),\ \forall \mathbf{y}, \mathbf{w} \in V$.

We briefly present here some fundamental properties of the trilinear functional $b(\cdot,\cdot,\cdot)$ defining the inertial operator B (see P.Constantin and C.Foias(11), R.Temam(12), V.Barbu(13)).

Proposition 1.1.1. *Let $1 \le N \le 3$. Then,*

$$b(\mathbf{y}, \mathbf{z}, \mathbf{w}) = -b(\mathbf{y}, \mathbf{w}, \mathbf{z}); \tag{1.1.9}$$

$$|b(\mathbf{y}, \mathbf{z}, \mathbf{w})| \le C\|\mathbf{y}\|_{m_1}\|\mathbf{z}\|_{m_2+1}\|\mathbf{w}\|_{m_3}, \tag{1.1.10}$$

where m_1, m_2, m_3 are positive numbers, satisfying:

$$\begin{cases} m_1 + m_2 + m_3 \ge \frac{N}{2}, & \text{if } m_i \ne \frac{N}{2}, \forall i \in \{1,2,3\}; \\ m_1 + m_2 + m_3 > \frac{N}{2}, & \text{if } \exists i \in \{1,2,3\}, m_i = \frac{N}{2}. \end{cases}$$

We note also the interpolation inequality:

$$\|\mathbf{y}\|_m \le C\|\mathbf{y}\|_l^{1-\alpha}\|\mathbf{y}\|_{l+1}^{\alpha}, \tag{1.1.11}$$

where $\alpha = m - l \in (0,1)$. Here $\|\cdot\|_m$ denotes the norm of the Sobolev space $H^m(\Omega)$.

Let $\mathbf{f}(t) = P\mathbf{f}_0(t)$ and $D \in L(U, H)$ be given by $D = PD_0$, where $P : (L^2(\Omega))^N \to H$ is the projection on H. Then we may rewrite the optimal control problem (P) as:

$$\textbf{(P)} \qquad \textbf{Min}\ \frac{1}{2}\int_0^T |\mathscr{C}(\mathbf{y}(t) - \mathbf{y}^0(t))|^2 dt + \int_0^T h(\mathbf{u}(t))dt;$$

subject to

$$\begin{cases} \mathbf{y}'(t) + \nu A\mathbf{y}(t) + B(\mathbf{y}(t)) = D\mathbf{u}(t) + \mathbf{f}(t), \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases} \tag{1.1.12}$$

with state constraint

$$\mathbf{y}(t) \in K, \qquad \forall t \in [0,T]. \tag{1.1.13}$$

The function $\mathbf{y} : [0,T] \to H$ is said to be weak solution to equation (1.1.12) if $\mathbf{y} \in \mathscr{Y}_w = L^2(0,T;V) \cap C_w(0,T;H) \cap W^{1,1}(0,T;V')$

$$\begin{cases} \frac{d}{dt}\langle \mathbf{y}(t), \Psi\rangle_{(V',V)} + \nu a(\mathbf{y},\Psi) + b(\mathbf{y},\mathbf{y},\Psi) = \langle \mathbf{f} + D\mathbf{u}, \Psi\rangle_{(V',V)} \ a.e. \ \text{in } (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \Psi \in V, \end{cases} \tag{1.1.14}$$

where $C_w(0,T;H)$ is the space of weak continuous functions $\mathbf{y} : [0,T] \to H$.

The function $\mathbf{y}$ is said to be strong solution to equation (1.1.12) if $\mathbf{y} \in W^{1,1}([0,T];H) \cap L^2(0,T;D(A))$, and (1.1.12) holds with $\frac{d\mathbf{y}}{dt} \in L^2(0,T;H)$.

The following hypothesis will be in effected throughout this chapter:

(i) $K \subset H$ is a closed convex subset with nonempty interior;

(ii) $\mathscr{C} \in L(V;H)$, $D \in L(U;H)$, $\mathbf{y}^0 \in L^2(0,T;H \cap D(\mathscr{C}^*\mathscr{C}))$, $\mathbf{f} \in L^2(0,T;H)$, $\mathbf{y}_0 \in V$;

(iii) $h : U \to (-\infty, +\infty]$ is a convex lower semicontinuous function. Moreover, there exist $\alpha > 0$ and $C \in \mathbb{R}$ such that

$$h(\mathbf{u}) \geq \alpha\|\mathbf{u}\|_U^2 + C, \ \forall \mathbf{u} \in U. \tag{1.1.15}$$

When we study problem (P) in the case that K is a closed convex subset of V, we need assumption

(ii') $\mathscr{C} \in L(V;H)$, $D \in L(U;V)$, $\mathbf{y}^0 \in L^2(0,T;H \cap D(\mathscr{C}^*\mathscr{C}))$, $\mathbf{f} \in L^2(0,T;H)$, $\mathbf{y}_0 \in V$.

1.2 Existence results

By admissible pair we mean $(\mathbf{y},\mathbf{u}) \in \mathscr{P}_w = \{(\mathbf{y},\mathbf{u}) \in \mathscr{Y}_w \times L^2(0,T;U); (\mathbf{y},\mathbf{u})$ solution to (1.1.14), $\mathbf{y}(t) \in K, \forall t \in [0,T]\}$. An optimal pair is an admissible pair which minimizes (P). To get the existence of optimal solution, we shall assume there exists at least one admissible pair.

Theorem 1.2.1. *The optimal control problem (P) has at least one optimal pair* $(\hat{\mathbf{y}}, \hat{\mathbf{u}})$*. In 2-D,* $\hat{\mathbf{y}}$ *is strong solution to equation (1.1.12).*

Remark 1.2.1. *When* $N = 3$*, if we assume that the admissible control set is a bounded subset of* $L^2(0,T;U)$*, then we can consider the strong solution in a local time interval* $(0,T^*)$*. By the similar method applied in the proof of Theorem 1.2.1, we can get the existence result, and the optimal state function* $\hat{\mathbf{y}} \in W^{1,2}(0,T^*;H) \cap L^2(0,T^*;D(A))$*. Moreover, the same result follows when the state constraint set* K *is a closed convex subset of* V*.*

1.3 The maximum principle

To get the maximum principle, we need to consider the strong solution of the Navier-Stokes equations. When $N = 3$, we need to consider the problem in such case with bounded admissible control set $\mathscr{U}_{ad} = \{\mathbf{u} \in L^2(0,T;U); \|D\mathbf{u}\|_{L^2(0,T;H)} \leq L\}$, and then we can consider the strong solution to Navier-Stokes equation in $(0, T^*)$, where $0 < T^* = T(L+\delta) < T(L)$. Here $\delta > 0$ is a fixed constant, and $T(L)$ is given by

$$T(L) = \frac{\nu}{3C_0^3[\|\mathbf{y}_0\|^2 + (\frac{2}{\nu})(\|\mathbf{f}\|^2_{L^2(0,T;H)} + L^2)]^3}. \tag{1.3.1}$$

Denote $\mathscr{D}(h) = \{\mathbf{u} \in L^2(0,T;U); \int_0^T h(\mathbf{u})dt < +\infty\}$. When $N = 3$, we shall assume that

$$\mathscr{D}(h) \subset \mathscr{U}_{ad}. \tag{1.3.2}$$

With this assumption, we can consider the strong solution in $[0, T^*]$ in 3-D without control constraint which is included in the definition of the function h inexplicitly.

Since in 2-D, the strong solution to equation (1.1.12) exists on arbitrary time interval $(0, T)$, such assumption is unnecessary. We still denote the interval $[0, T^*]$ where assumption (1.3.2) holds by $[0, T]$.

We need also the following assumption:

(iv) There exists $(\tilde{\mathbf{z}}, \tilde{\mathbf{u}}) \in C(0,T;H) \times L^2(0,T;U)$ solution to equation

$$\begin{cases} \tilde{\mathbf{z}}'(t) + \nu A\tilde{\mathbf{z}} + B'(\mathbf{y}^*)\tilde{\mathbf{z}} = B(\mathbf{y}^*) + D\tilde{\mathbf{u}}(t) + \mathbf{f}(t), \\ \tilde{\mathbf{z}}(0) = \mathbf{y}_0. \end{cases} \tag{1.3.3}$$

such that $\tilde{\mathbf{z}}(t) \in \operatorname{int}K$, for t in a dense subset of $[0, T]$.

Here $\mathbf{y}^*$ is the optimal state function for the optimal control problem (P). $B'(\mathbf{y}^*)$ is the operator defined by

$$\langle B'(\mathbf{y}^*)z, w\rangle = b(\mathbf{y}^*, \mathbf{z}, \mathbf{w}) + b(\mathbf{z}, \mathbf{y}^*, \mathbf{w}).$$

Inasmuch as

$$B(\mathbf{y}^*) \in L^2(0,T;H),\ |(B'(\mathbf{y}^*)\mathbf{z}, \mathbf{z})| \leq \frac{\nu}{4}\|\mathbf{z}\|^2 + C_\nu|\mathbf{z}|^2,$$

we know that equation (1.3.3) has a solution $\tilde{\mathbf{z}} \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A))$.

Theorem 1.3.1. *Let $(\mathbf{y}^*(t), \mathbf{u}^*(t))$ be the optimal pair for the optimal control problem (P). Then under assumptions (i)∼(iv), there are $\mathbf{p} \in L^\infty(0,T;H)$ and $\boldsymbol{\omega} \in BV([0,T];H)$, such that:*

$$D^*\mathbf{p}(t) \in \partial h(\mathbf{u}^*(t)) \quad a.e. \text{ in } [0,T], \tag{1.3.4}$$

$$\mathbf{p}(t) = -\int_t^T U(s,t)(\mathscr{C}^*\mathscr{C}(\mathbf{y}^*(s) - \mathbf{y}^0(s)))ds - \int_t^T U(s,t)d\boldsymbol{\omega}(s), \tag{1.3.5}$$

and

$$\int_0^T \langle d\boldsymbol{\omega}(t), \mathbf{y}^*(t) - \mathbf{x}(t)\rangle \geq 0, \forall \mathbf{x} \in \mathcal{K}. \tag{1.3.6}$$

Here D^*, $\mathscr{C}^*$, $B'(\mathbf{y}^*(t))^*$, are the adjoint operators of D, $\mathscr{C}$ and $B'(\mathbf{y}^*(t))$ respectively, $U(s,t)$ is the evolution operator generated by the operator $\nu A + B'(\mathbf{y}^*(t))^*$. We recognize in (1.3.5) the mild form of the dual equation

$$\begin{cases} \mathbf{p}'(t) = \nu Ap(t) + B'(\mathbf{y}^*)^*\mathbf{p}(t) + \mathscr{C}^*\mathscr{C}(\mathbf{y}^*(t) - \mathbf{y}^0(t)) + \mu_{\boldsymbol{\omega}}(t), a.e. \text{ in } (0,T), \\ \mathbf{p}(T) = 0. \end{cases} \tag{1.3.7}$$

Theorem 1.3.2 below is the analogue of Theorem 1.3.1 under the weaker assumption :
(v) K is a closed convex subset of V, and there exists $(\tilde{\mathbf{z}}, \tilde{\mathbf{u}}) \in C(0,T;H) \times L^2(0,T;U)$ solution to equation (1.3.3), such that $\tilde{\mathbf{z}}(t) \in \mathrm{int}_V K$, for t in a dense subset of $[0,T]$.

Here $\mathrm{int}_V K$ is the interior of K with respect to topology of V.

Theorem 1.3.2. *Let $(\mathbf{y}^*(t), \mathbf{u}^*(t))$ be the solution for optimal control problem (P). Then under assumptions (ii'),(iii),(v), there are $\mathbf{p} \in L^\infty(0,T;V'), \boldsymbol{\omega} \in BV([0,T];V')$, such that (1.3.4) and (1.3.5) hold, and (1.3.6) hold in the sense of*

$$\int_0^T \langle d\boldsymbol{\omega}(t), \mathbf{y}^*(t) - \mathbf{x}(t)\rangle_{(V',V)} \geq 0, \forall \mathbf{x} \in \mathcal{K}. \tag{1.3.8}$$

We shall consider the reflexive Banach space E as H or V, and denote by $(\cdot,\cdot)$ the dual product between E and it's dual of E (When $E = H$, it is the scalar product in H), by $\|\cdot\|$ the norm of E. Under the hypothesis of Theorem 1.3.1 or the hypothesis of Theorem 1.3.2, We give a corollary here:

Corollary 1.3.1. *Let the pair $(\mathbf{y}^*, \mathbf{u}^*)$ be the optimal pair in problem (P), then there exist $\boldsymbol{\omega} \in BV([0,T];E')$ and $\mathbf{p}$ satisfying along with $\mathbf{y}^*, \mathbf{u}^*$, equations (1.3.4), (1.3.5), (1.3.6) (or(1.3.8)) and*

$$\boldsymbol{\omega}_a(t) \in N_K(\mathbf{y}^*(t)), a.e. t \in (0,T), \tag{1.3.9}$$

$$d\boldsymbol{\omega}_s \in \mathscr{N}_{\mathcal{K}}(\mathbf{y}^*). \tag{1.3.10}$$

Here $\boldsymbol{\omega}_a(t)$ is the weak derivative of $\boldsymbol{\omega}(t)$, and $d\boldsymbol{\omega}_s$ is the singular part of measure $d\boldsymbol{\omega}$. $N_K(\mathbf{y}^*(t))$ is the normal cone to K at $\mathbf{y}^*(t)$, and $\mathscr{N}_{\mathcal{K}}(\mathbf{y}^*)$ is the normal cone to $\mathcal{K}$ at $\mathbf{y}^*$ which is defined by

$$\mathcal{N}_{\mathcal{K}}(\mathbf{y}) = \{\mu \in M(0,T;E'); \mu(\mathbf{y} - \mathbf{x}) \geq 0, \forall \mathbf{x} \in \mathcal{K}\} \tag{1.3.11}$$

1.4 Examples

In this section, we shall give some applications of the above results in some special cases of state constraints wherein Theorem 1.3.1 and Theorem 1.3.2 can be applied.

Example 1.4.1. *Let K be the set $K = \{\mathbf{y} \in H; \int_\Omega |\mathbf{y}(\mathbf{x})|^2 d\mathbf{x} \leq \rho^2\}$. Then K is a closed convex set in H. Since*

$$\|\tilde{\mathbf{z}}\|_{C([0,T];H)} \leq C(\|B(\mathbf{y}^*) + D\tilde{\mathbf{u}} + \mathbf{f}\|_{L^2(0,T;H)})),$$

we may apply Theorem 1.3.1 to get the necessary condition of the optimal control pair after checking whether condition (iv) is satisfied or not. The set K physically gives a constraint on the turbulence kinetic energy. In this case, the maximum principle can be described as following:

$$D^*\mathbf{p}(t) \in \partial h(\mathbf{u}^*(t)) \quad \text{a.e. in } [0,T], \tag{1.4.1}$$

$$\begin{cases} \mathbf{p}_1'(t) = \nu A\mathbf{p}_1(t) + (B'(\mathbf{y}^*(t))^*)\mathbf{p}_1(t) + \mathscr{C}^*\mathscr{C}(\mathbf{y}^*(t) - \mathbf{y}^0(t)) + \boldsymbol{\omega}_a(t), \ a.e. \text{ in } (0,T), \\ \mathbf{p}_1(T) = 0. \end{cases} \tag{1.4.2}$$

$$\begin{cases} \mathbf{p}_2'(t) = \nu A\mathbf{p}_2(t) + (B'(\mathbf{y}^*(t))^*)\mathbf{p}_2(t) + d\boldsymbol{\omega}_s, \ \ a.e. \text{ in } (0,T), \\ \mathbf{p}_2(T) = 0. \end{cases} \tag{1.4.3}$$

Moreover,

$$\boldsymbol{\omega}_a(t) \in N_K(\mathbf{y}^*(t)) = \{\lambda(t)\mathbf{y}^*(t); \lambda(t) \geq 0, a.e. \text{ in } (0,T)\}. \tag{1.4.4}$$

Here $\mathbf{p}_1, \mathbf{p}_2$ is the decomposition of $\mathbf{p}$, that is $\mathbf{p}(t) = \mathbf{p}_1(t) + \mathbf{p}_2(t)$. Since $\boldsymbol{\omega}_a \in L^2(0,T;H)$, $d\boldsymbol{\omega}_s$ is the singular part of the measure $d\boldsymbol{\omega}$, we know that equation (1.4.2) has a strong solution $\mathbf{p}_1 \in C([0,T];H)$, while equation (1.4.3) has only a mild solution $\mathbf{p}_2(t) = -\int_t^T U(s,t)d\boldsymbol{\omega}_s$.

Example 1.4.2. *Let K be the so called Enstrophy set*

$$K = \{\mathbf{y} \in V; \int_\Omega |\nabla \times \mathbf{y}|^2 d\mathbf{x} \leq \rho^2\},$$

where $\nabla \times \mathbf{y} = curl\ \mathbf{y}(\mathbf{x})$. It is true that the norm $|\nabla \times \mathbf{y}|$ is equivalent to the norm $\|y\|$ in the space V. In fluid mechanics, the enstrophy $\mathcal{E}(\mathbf{y}) = \int_\Omega |\nabla \times \mathbf{y}|^2 d\mathbf{x}$ can be interpreted as another type of potential density. More precisely, the quantity directly related to the kinetic energy in the flow model that corresponds to dissipation effects in the fluid. It is particularly useful in the study of turbulent flows, and it is often identified in the study of trusters, as well as the flame field. Enstrophy set gives a constraint on the vorcity of the fluid motion. Since

$$\|\tilde{\mathbf{z}}\|_{C([0,T];V)} \leq C(\|B(\mathbf{y}^*) + D\tilde{\mathbf{u}} + \mathbf{f}\|_{L^2(0,T;H)}),$$

we may apply Theorem 1.3.2 to get the necessary condition of the optimal control pair after checking whether condition (v) is satisfied or not. In this case, the maximum principle can be described by (1.4.1), (1.4.2) and (1.4.3). Moreover,

$$\boldsymbol{\omega}_a(t) \in N_K(\mathbf{y}^*(t)) = \{\lambda(t)A\mathbf{y}^*(t); \lambda(t) \geq 0, a.e. \text{ in } (0,T)\}.$$

Example 1.4.3. *Let K be the so called Helicity set,*

$$K = \{\mathbf{y} \in V; \int_\Omega \langle \mathbf{y}, curl\ \mathbf{y}\rangle^2 d\mathbf{x} + \lambda \int_\Omega |\nabla \mathbf{y}|^2 d\mathbf{x} \leq \rho^2\},$$

where λ, ρ are positive constants. The helicity set plays an important role in fluid mechanics, and in particular, it is an invariant set of Euler's equation for incompressible fluids(See (14)). By the same argument as in Example 1.4.2, we know that it is feasible to apply Theorem 1.3.2 to get the necessary condition of the optimal pair when the state constrained set is Helicity set, and in this case, the maximum principle can be described by (1.4.1), (1.4.2) and (1.4.3). Moreover,

$$\boldsymbol{\omega}_a(t) \in N_K(\mathbf{y}^*(t)) = \{\lambda(t)(A\mathbf{y}^*(t) + \text{curl}\mathbf{y}^*(t)); \lambda(t) \geq 0, a.e. \text{ in } (0,T)\}.$$

2

Optimal control problems with state constraint governed by MHD equations

2.1 Introduction

In this chapter, we shall study the optimal control problem

$$(\mathbf{P}) \quad \textbf{Minimize} \ \frac{1}{2}\int_0^T \left(\int_\Omega |\mathscr{C}(\mathbf{y}(t,\mathbf{x}) - \mathbf{y}^0(t,\mathbf{x}))|^2\right) d\mathbf{x}dt + \int_0^T h(\mathbf{u}(t))dt;$$

subject to the magnetohydrodynamic (MHD) equation

$$\begin{cases} \frac{\partial \mathbf{y}}{\partial t} - \nu\triangle\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla(\frac{\mathbf{B}^2}{2}) - (\mathbf{B}\cdot\nabla)\mathbf{B} + \nabla p = D_0\mathbf{u} + \mathbf{f}_0, & \text{in } \Omega\times(0,T), \\ \frac{\partial \mathbf{B}}{\partial t} - \eta curl(curl\mathbf{B}) + (\mathbf{y}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{y} = \mathbf{g}, & \text{in } \Omega\times(0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \mathbf{B}(0) = \mathbf{B}_0, & \text{in } \Omega, \\ \nabla\cdot\mathbf{y} = 0, \nabla\cdot\mathbf{B} = 0, & \text{in } \Omega\times(0,T), \\ \mathbf{y} = 0, \mathbf{B}\cdot\mathbf{n} = 0, (curl\mathbf{B})\times\mathbf{n} = 0, & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{2.1.1}$$

with state constraint

$$\mathbf{y}(t) \in K, \qquad \forall t \in (0,T), \tag{2.1.2}$$

where K is a closed convex subset in H. Here Ω is a bounded and open subset of $\mathbb{R}^N$ ($N = 2,3$) with smooth boundary $\partial\Omega$, $T > 0$ is a given constant, $\nu > 0$ is the viscosity constant, $\mathbf{f}_0 \in L^2(0,T;(L^2(\Omega))^N)$ is a source field, $\mathbf{y}(\mathbf{x},t) = (y_1(\mathbf{x},t),\cdots,y_N(\mathbf{x},t))$ is the velocity vector, p stands for the pressure, $D_0 \in L(U;(L^2(\Omega))^N)$, and $\mathbf{u} \in L^2(0,T;U)$, where U is a Hilbert space. $\mathbf{B}(\mathbf{x},t) = (B_1(\mathbf{x},t),\cdots,B_N(\mathbf{x},t))$ is the magnetic field, $\eta > 0$ is the magnetic resistivity. The function $\mathbf{g} \in L^2(0,T;(L^2(\Omega))^N)$ and $\text{div}\mathbf{g} = 0$.

The function $h : U \to (-\infty,+\infty]$ is convex and lower semicontinuous, $\mathbf{y}^0 \in L^2(0,T;H)$, and $\mathscr{C} \in L(V,H)$. As in Chapter 1, the same two cases of physical interest are covered by the cost functional of this form.

When $N = 2$, the MHD equation is the modification of equations (2.1.1) by substituting the term $curl(curl\mathbf{B})$ by $\tilde{curl}(curl\mathbf{B})$ and the boudary condition $curl\mathbf{B} \times \mathbf{n} = 0$ by $curl\mathbf{B} = 0$, where

$$curl\mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \forall \mathbf{u} = (u_1, u_2),$$
$$\tilde{curl}\phi = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}), \text{for every scalar function } \phi,$$

and we have the two dimensional formula $\tilde{curl}(curl\mathbf{B}) = grad\ div\mathbf{B} - \triangle\mathbf{B}$.

Let us introduce some functional spaces and some operators to represent the MHD equation (2.1.1) as infinite dimensional differential equations.

Define the Hilbert space V_1 by

$$V_1 = \{\mathbf{B} \in (H^1(\Omega))^N; div\mathbf{B} = 0 \text{ in } \Omega \text{ and } \mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Let $P : (L^2(\Omega))^N \to H$ be the Leray projection, and let

$$A_1(\mathbf{B}) = curl(curl\mathbf{B}) = -\triangle\mathbf{B}, \forall \mathbf{B} \in D(A_1) = \{B \in (H^2(\Omega))^N \cap V_1; curl\mathbf{B} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

We note that A and A_1 are self-adjoint on H.

We endow the Hilbert space V_1 with the scalar product

$$\langle \mathbf{B}, \mathbf{C} \rangle_1 = \sum_{i=1}^{N} \int_{\Omega} curlB_i \cdot curlC_i dx,\ \forall \mathbf{B}, \mathbf{C} \in V_1.$$

and the norm induced by this scalar products is equivalent to the norm in $(H_0^1(\Omega))^N$, which will be denoted by $\|\cdot\|$. The dual space of V_1 will be denoted by V_1'. If there is no ambiguous, we shall denote by $\langle \cdot, \cdot \rangle$ the dual product between V_1 and V_1'. We define now the operators $\mathscr{B}_1 : V \to V', \mathscr{B}_2 : V_1 \to V', \mathscr{B}_3 : V \times V_1 \to V_1', \mathscr{B}_4 : V_1 \times V \to V_1'$, by

$$\langle \mathscr{B}_1(\mathbf{y}), \mathbf{w} \rangle = b(\mathbf{y}, \mathbf{y}, \mathbf{w}),\ \forall \mathbf{w} \in V,$$
$$\langle \mathscr{B}_2(\mathbf{B}), \mathbf{w} \rangle = -b(\mathbf{B}, \mathbf{B}, \mathbf{w}),\ \forall \mathbf{w} \in V,$$
$$\mathscr{B}_3(\mathbf{y}, \mathbf{B}) = (\mathbf{y} \cdot \nabla)\mathbf{B},\ \forall \mathbf{y} \in V, \mathbf{B} \in V_1,$$
$$\mathscr{B}_4(\mathbf{B}, \mathbf{y}) = -(\mathbf{B} \cdot \nabla)\mathbf{y},\ \forall \mathbf{y} \in V, \mathbf{B} \in V_1.$$

Let $\mathbf{f}(t) = P\mathbf{f}_0(t)$ and $D \in L(U, H)$ be given by $D = PD_0$. Denote $\mathbb{A} = A \times A_1, D(\mathbb{A}) = D(A) \times D(A_1), \mathbb{V} = V \times V_1, \mathbb{H} = H \times H$. In the following, if there is no ambiguity, we shall still denote by $|\cdot|$ and $\|\cdot\|$ the norms of space $\mathbb{H}$ and space $\mathbb{V}$, respectively. Then we may rewrite the optimal control problem (P) as:

$$\textbf{(P)} \qquad \textbf{Min}\ \frac{1}{2}\int_0^T |\mathscr{C}(\mathbf{y}(t) - \mathbf{y}^0(t))|^2 + \int_0^T h(\mathbf{u}(t))dt;$$

subject to

$$\begin{cases} \mathbf{y}'(t) + \nu A\mathbf{y}(t) + \mathscr{B}_1(\mathbf{y}(t)) + \mathscr{B}_2(\mathbf{B}(t)) = D\mathbf{u}(t) + \mathbf{f}(t), \\ \mathbf{B}'(t) + \eta A_1\mathbf{B}(t) + \mathscr{B}_3(\mathbf{y}(t), \mathbf{B}(t)) + \mathscr{B}_4(\mathbf{B}(t), \mathbf{y}(t)) = \mathbf{g}(t), \\ \mathbf{y}(0) = \mathbf{y}_0, \mathbf{B}(0) = \mathbf{B}_0. \end{cases} \tag{2.1.3}$$

with state constraint

$$\mathbf{y}(t) \in K \qquad \forall t \in [0, T]. \tag{2.1.4}$$

Similar to the arguments in Refs. (15), (13) and (12), for each $\mathbf{f}, \mathbf{g}, D\mathbf{u} \in L^2(0, T; H)$ and $(\mathbf{y}_0, \mathbf{B}_0) \in \mathbb{V}$, equation (2.1.3) has a unique solution $(\mathbf{y}, \mathbf{B}) \in W^{1,2}(0, T; \mathbb{H}) \cap L^2(0, T; D(\mathbb{A}))$ when $N = 2$, while in the case $N = 3$, for each $\mathbf{u} \in L^2(0, T; U)$, there exists $0 < T(\mathbf{u}) \leq T$ such that (2.1.3) has a unique solution $(\mathbf{y}(\cdot; \mathbf{u}), \mathbf{B}(\cdot; \mathbf{u})) \in W^{1,2}(0, T^*; \mathbb{H}) \cap L^2(0, T^*; D(\mathbb{A}))$ for all $T^* < T(\mathbf{u})$. Here $T(\mathbf{u})$ is given by

$$T(\mathbf{u}) = \frac{1}{C_0[\|\mathbf{y}_0\|^2 + \|\mathbf{B}_0\|^2 + (\frac{1}{\nu})\|\mathbf{f} + D\mathbf{u}\|^2_{L^2(0,T;H)} + (\frac{1}{\eta})\|\mathbf{g}\|^2_{L^2(0,T;H)}]^3}, \tag{2.1.5}$$

where C_0 is a positive constant independent of $\mathbf{y}_0, \mathbf{B}_0, \mathbf{u}$, ν and η.

Another way to formulate the control problem is in the framework of weak solutions to equation (2.1.3), that is $(\mathbf{y}, \mathbf{B}) \in \mathscr{Y}_w = L^2(0, T; \mathbb{V}) \cap C_w(0, T; \mathbb{H}) \cap W^{1,1}(0, T; \mathbb{V}')$, satisfying , for each $\mathbf{z} \in V, \mathbf{C} \in V_1$

$$\begin{cases} \langle \mathbf{y}'(t), \mathbf{z}\rangle_{(V',V)} + \nu a(\mathbf{y}, \mathbf{z}) + b(\mathbf{y}, \mathbf{y}, \mathbf{z}) - b(\mathbf{B}, \mathbf{B}, \mathbf{z}) = \langle D\mathbf{u} + \mathbf{f}, \mathbf{z}\rangle_{(V',V)}, \text{a.e. in } (0, T), \\ \langle \mathbf{B}'(t), \mathbf{C}\rangle_{(V_1',V_1)} + \eta\langle \mathbf{B}, \mathbf{C}\rangle_1 + b(\mathbf{y}, \mathbf{B}, \mathbf{C}) - b(\mathbf{B}, \mathbf{y}, \mathbf{C}) = \langle \mathbf{g}, \mathbf{C}\rangle_{(V_1',V_1)}, \text{ a.e. in } (0, T), \\ \mathbf{y}(0) = \mathbf{y}_0, \mathbf{B}(0) = \mathbf{B}_0. \end{cases} \tag{2.1.6}$$

where $C_w(0, T; \mathbb{H})$ is the space of weak continuous functions $\Upsilon : [0, T] \to \mathbb{H}$. It is known that there exists at least a weak solution to equation (2.1.3) for each $\mathbf{u} \in L^2(0, T; U)$ (see (13) pp.265, Th.5.12 and (15)). We shall denote $\mathscr{P}_w = \{((\mathbf{y}, \mathbf{B}), \mathbf{u}) \in \mathscr{Y}_w \times L^2(0, T; U); ((\mathbf{y}, \mathbf{B}), \mathbf{u})$ solution to (2.1.6), $\mathbf{y}(t) \in K, \forall t \in [0, T]\}$.

The following hypothesis will be in effected throughout this chapter:

(i) $K \subset H$ is a closed convex subset with nonempty interior;

(ii) $\mathscr{C} \in L(V; H)$, $D \in L(U; H)$, $\mathbf{y}^0 \in L^2(0, T; H \cap D(\mathscr{C}^*\mathscr{C}))$, $\mathbf{f}, \mathbf{g} \in L^2(0, T; H)$, $(\mathbf{y}_0, \mathbf{B}_0) \in \mathbb{V}$;

(iii) $h : U \to (-\infty, +\infty]$ is a convex lower semicontinuous function. Moreover, there exist $\alpha > 0$ and $C \in \mathbb{R}$ such that

$$h(\mathbf{u}) \geq \alpha\|\mathbf{u}\|^2_U + C, \ \forall \mathbf{u} \in U. \tag{2.1.7}$$

When we study problem (P) in the case that K is a closed convex subset of V, we need assumption (ii') which is assumption (ii) together with the assumption $D \in L(U; V)$.

2.2 Existence results

By admissible pair we mean $((\mathbf{y}, \mathbf{B}), \mathbf{u}) \in \mathscr{P}_w$, which satisfies equation (2.1.3) in the weak sense, i.e. (2.1.6). An optimal pair is an admissible pair which minimizes (P). To get the existence of optimal solution, we assume there exists at least one admissible pair.

Theorem 2.2.1. *The optimal control problem (P) has at least one optimal pair* $((\hat{\mathbf{y}}, \hat{\mathbf{B}}), \hat{\mathbf{u}})$. *In 2-D,* $(\hat{\mathbf{y}}, \hat{\mathbf{B}})$ *is strong solution to equation 2.1.3.*

Remark 2.2.1. *When* $N = 3$, *if we assume that the admissible control set is a bounded subset of* $L^2(0, T; U)$, *then we can consider the strong solution in a local time interval* $(0, T^*)$. *By the similar method applied in the proof of Theorem 2.2.1, we can get the existence result, and the optimal state function* $(\hat{\mathbf{y}}, \hat{\mathbf{B}}) \in W^{1,2}(0, T^*; \mathbb{H}) \cap L^2(0, T^*; D(\mathbb{A}))$. *Moreover, the same result follows when the state constraint set* K *is a closed convex subset of* V.

2.3 The maximum principle

To get the maximum principle, we need to consider the strong solution of the MHD equations. As we mentioned, when $N = 3$, we need to consider the problem of such case with bounded admissible control set as

$$\mathscr{U}_{ad} = \{\mathbf{u} \in L^2(0, T; U); \|D\mathbf{u}\|_{L^2(0,T;H)} \leq L\},$$

and then we can consider the strong solution to MHD equation in $(0, T^*)$, where $0 < T^* = T(L + \delta) < T(L)$. Here $\delta > 0$ is a fixed constant, and $T(L)$ is given by (2.1.5), i.e.

$$T(L) = \frac{1}{C_0[\|\mathbf{y}_0\|^2 + \|\mathbf{B}_0\|^2 + (\frac{2}{\nu})(\|\mathbf{f}\|^2_{L^2(0,T;H)} + L^2) + (\frac{1}{\eta})\|\mathbf{g}\|^2_{L^2(0,T;H)}]^3}. \tag{2.3.1}$$

Denote $\mathscr{D}(h) = \{\mathbf{u} \in L^2(0, T; U); \int_0^T h(\mathbf{u})dt < +\infty\}$. When $N = 3$, we shall assume that

$$\mathscr{D}(h) \subset \mathscr{U}_{ad}. \tag{2.3.2}$$

With this assumption, we can consider the strong solution in $[0, T^*]$ in 3-D without control constraint which is included in the definition of the function h inexplicitly. We have given an example of function h to show that this assumption can be easily fullfilled.

Since in 2-D, the strong solution to equation (2.1.3) exists on arbitrary time interval $(0, T)$, such assumption is unnecessary. We still denote the interval $[0, T^*]$ where assumption (2.3.2) holds by $[0, T]$.

We need also the following assumption:

(iv) There exists $((\tilde{\mathbf{z}}, \tilde{\mathbf{E}}), \tilde{\mathbf{u}}) \in C(0, T; \mathbb{H}) \times L^2(0, T; U)$ solution to equation

$$\begin{cases} \tilde{\mathbf{z}}'(t) + \nu A\tilde{\mathbf{z}} + (\mathscr{B}_1'(\mathbf{y}^*))\tilde{\mathbf{z}} + (\mathscr{B}_2'(\mathbf{B}^*))\tilde{\mathbf{E}} = \mathscr{B}_1(\mathbf{y}^*) + \mathscr{B}_2(\mathbf{B}^*) + D\tilde{\mathbf{u}} + \mathbf{f}, \\ \tilde{\mathbf{E}}'(t) + \eta A_1\tilde{\mathbf{E}} + (\mathscr{B}_4'(\mathbf{B}^*))\tilde{\mathbf{z}} + (\mathscr{B}_3'(\mathbf{y}^*))\tilde{\mathbf{E}} = \mathscr{B}_4'(\mathbf{B}^*)\mathbf{y}^*, \\ \tilde{\mathbf{z}}(0) = \mathbf{y}_0, \tilde{\mathbf{E}}(0) = \mathbf{B}_0, \end{cases} \tag{2.3.3}$$

such that $\tilde{\mathbf{z}}(t) \in \mathrm{int}K$, for t in a dense subset of $[0, T]$.

Here $(\mathbf{y}^*, \mathbf{B}^*)$ is the optimal state function for the optimal control problem (P). $\mathscr{B}_1'(\mathbf{y}^*)$, $\mathscr{B}_2'(\mathbf{B}^*), \mathscr{B}_3'(\mathbf{y}^*), \mathscr{B}_4'(\mathbf{B}^*)$ are the operators defined by

$$\begin{aligned} \langle \mathscr{B}_1'(\mathbf{y}^*)\mathbf{z}, \mathbf{w}\rangle &= b(\mathbf{y}^*, \mathbf{z}, \mathbf{w}) + b(\mathbf{z}, \mathbf{y}^*, \mathbf{w}), \\ \langle \mathscr{B}_2'(\mathbf{B}^*)\mathbf{E}, \mathbf{w}\rangle &= -b(\mathbf{B}^*, \mathbf{E}, \mathbf{w}) - b(\mathbf{E}, \mathbf{B}^*, \mathbf{w}), \\ \langle \mathscr{B}_3'(\mathbf{y}^*)\mathbf{E}, \mathbf{w}\rangle &= b(\mathbf{y}^*, \mathbf{E}, \mathbf{w}) - b(\mathbf{E}, \mathbf{y}^*, \mathbf{w}), \\ \langle \mathscr{B}_4'(\mathbf{B}^*)\mathbf{z}, \mathbf{w}\rangle &= b(\mathbf{z}, \mathbf{B}^*, \mathbf{w}) - b(\mathbf{B}^*, \mathbf{z}, \mathbf{w}). \end{aligned}$$

Theorem 2.3.1. *Let* $((\mathbf{y}^*(t), \mathbf{B}^*(t)), \mathbf{u}^*(t))$ *be the optimal pair for the optimal control problem* (P). *Then under assumptions (i)∼(iv), there are* $(\mathbf{p}, \mathbf{q}) \in L^\infty(0, T; \mathbb{H})$ *and* $\boldsymbol{\omega} \in BV([0, T]; H)$*, such that:*

$$D^*\mathbf{p}(t) \in \partial h(\mathbf{u}^*(t)) \quad a.e. \ \text{ in } [0, T], \tag{2.3.4}$$

$$\begin{pmatrix} \mathbf{p}(t) \\ \mathbf{q}(t) \end{pmatrix} = -\int_t^T U(s,t) \begin{pmatrix} \mathscr{C}^*\mathscr{C}(\mathbf{y}^*(s) - \mathbf{y}^0(s)) \\ 0 \end{pmatrix} ds - \int_t^T U(s,t) \begin{pmatrix} d\boldsymbol{\omega}(s) \\ 0 \end{pmatrix}, \tag{2.3.5}$$

and

$$\int_0^T \langle d\boldsymbol{\omega}(t), \mathbf{y}^*(t) - \mathbf{x}(t) \rangle \geq 0, \forall \mathbf{x} \in \mathcal{K}. \tag{2.3.6}$$

Here D^*, $\mathscr{C}^*$ are the adjoint operators of D, $\mathscr{C}$ respectively, $U(s,t)$ is the evolution operator generated by the operator $\begin{pmatrix} \nu A & 0 \\ 0 & \eta A_1 \end{pmatrix} + \begin{pmatrix} \mathscr{B}_1'(\mathbf{y}^*)^* & \mathscr{B}_4'(\mathbf{B}^*)^* \\ \mathscr{B}_2'(\mathbf{B}^*)^* & \mathscr{B}_3'(\mathbf{y}^*)^* \end{pmatrix}$. We recognize in (2.3.5) the mild form of the dual equation

$$\begin{cases} \mathbf{p}'(t) = \nu A\mathbf{p}(t) + \mathscr{B}_1'(\mathbf{y}^*)^*\mathbf{p}(t) + \mathscr{B}_4'(B^*)^*\mathbf{q}(t) + \mathscr{C}^*\mathscr{C}(\mathbf{y}^*(t) - \mathbf{y}^0(t)) + \mu_{\boldsymbol{\omega}}(t), \\ \mathbf{q}'(t) = \eta A_1\mathbf{q}(t) + \mathscr{B}_2'(\mathbf{B}^*)^*\mathbf{p}(t) + \mathscr{B}_3'(\mathbf{y}^*)^*\mathbf{q}(t), \ a.e. \text{ in } (0, T), \\ \mathbf{p}(T) = 0, \mathbf{q}(T) = 0. \end{cases} \tag{2.3.7}$$

Here $\mathscr{B}_i'(\boldsymbol{\xi})^*$ is the adjoint operator of $\mathscr{B}_i'(\boldsymbol{\xi})$, $i = 1, 2, 3, 4, \boldsymbol{\xi} = \mathbf{B}^*$ or $\mathbf{y}^*$.
Theorem 2.3.2 below is the analogue of Theorem 2.3.1 under the weaker assumption :
(v) K is a closed convex subset of V, and there are $((\tilde{\mathbf{z}}, \tilde{\mathbf{B}}), \tilde{\mathbf{u}}) \in C(0, T; \mathbb{H}) \times L^2(0, T; U)$ solution to equation (2.3.3), such that $\tilde{\mathbf{z}}(t) \in \text{int}_V K$, for t in a dense subset of $[0, T]$.

Here $\text{int}_V K$ is the interior of K with respect to topology of V.

Theorem 2.3.2. *Let* $((\mathbf{y}^*(t), \mathbf{B}^*(t)), \mathbf{u}^*(t))$ *be the solution for optimal control problem* (P). *Then under assumptions (ii'),(iii),(v), there are* $(\mathbf{p}, \mathbf{q}) \in L^\infty(0, T; \mathbb{V}'), \boldsymbol{\omega} \in BV([0, T]; V')$*, such that (2.3.4) and (2.3.5) hold, and (2.3.6) holds in the sense of*

$$\int_0^T \langle d\boldsymbol{\omega}(t), \mathbf{y}^*(t) - \mathbf{x}(t) \rangle_{(V', V)} \geq 0, \forall \mathbf{x} \in \mathcal{K}. \tag{2.3.8}$$

We shall consider the reflexive Banach space E as H or V, and denote by $(\cdot, \cdot)$ the dual product between E and it's dual of E (When $E = H$, it is the scalar product in H), by $\| \cdot \|$ the norm of E. Under the hypothesis of Theorem 2.3.1 or the hypothesis of Theorem 2.3.2, We give a corollary here:

Corollary 2.3.1. *Let the pair* $((\mathbf{y}^*, \mathbf{B}^*), \mathbf{u}^*)$ *be the optimal pair in problem (P), then there exist* $\boldsymbol{\omega} \in BV([0, T]; E')$ *and* $(\mathbf{p}, \mathbf{q}) \in L^\infty(0, T; E')$ *satisfying along with* $((\mathbf{y}^*, \mathbf{B}^*), \mathbf{u}^*)$*, equations (2.3.4),(2.3.5),(2.3.6) (or(2.3.8)) and*

$$\boldsymbol{\omega}_a(t) \in N_K(\mathbf{y}^*(t)), a.e. \text{ in } (0, T), \tag{2.3.9}$$

$$d\boldsymbol{\omega}_s \in \mathscr{N}_{\mathcal{K}}(\mathbf{y}^*). \tag{2.3.10}$$

Here $\boldsymbol{\omega}_a(t)$ is the weak derivative of $\boldsymbol{\omega}(t)$, and $d\omega_s$ is the singular part of measure $d\boldsymbol{\omega}$. $N_K(\mathbf{y}^*(t))$ is the normal cone to K at $\mathbf{y}^*(t)$, and $\mathscr{N}_{\mathcal{K}}(\mathbf{y}^*)$ is the normal cone to $\mathcal{K}$ at $\mathbf{y}^*$ which is precised in 1.3.11 (See the proof of Corollary 1.3.1).

Similarly, we can apply the maximum principles obtained in this Section to the examples in the last Chapter.

3

Boundary optimal control of time-periodic Stokes-Oseen flows

3.1 Formulation of the Problem

Here we shall consider the optimal control problem :

$$\text{(P)} \qquad \text{Min}\{\frac{1}{2}\int_0^T\int_\Omega |\nabla(\mathbf{y}-\mathbf{y}^0)|^2 d\mathbf{x}dt + \frac{1}{2}\int_0^T\int_{\partial\Omega}|\mathbf{u}|^2 d\mathbf{x}dt\}$$

subject to the periodic Stokes-Oseen equations with Dirichlet boundary condition:

$$\begin{cases} \frac{d\mathbf{y}}{dt} - \nu\triangle\mathbf{y} + (\mathbf{f}_1\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{f}_2 + \nabla p = \mathbf{f}_0 & \text{in } \Omega\times(0,T), \\ \mathbf{y}(0) = \mathbf{y}(T) & \text{in } \Omega, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } \Omega\times(0,T), \\ \mathbf{y} = \mathbf{u} & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{3.1.1}$$

Here, Ω is a bounded open subset with smooth boundary in $\mathbb{R}^2$. Function $\mathbf{u}(t) \in U, \forall t \in (0,T)$ is the boundary control, where $U = \{\mathbf{u} \in (L^2(\partial\Omega))^2; \mathbf{u}\cdot\mathbf{n} = 0\}$. Here, $\mathbf{n}$ is the outer normal vector of $\partial\Omega$. The function $\mathbf{y}^0 \in L^2(0,T;(H^1(\Omega))^2)$ is the objective velocity field. The functions $\mathbf{f}_i(\mathbf{x}), i = 1,2$ are steady state functions and $\mathbf{f}_i \in (W^{2,\infty}(\Omega))^2 \cap H$.

Define the spaces

$$V^s := (H^s(\Omega))^2 \cap H, s \in \mathbb{R},$$

where $H^s(\Omega)$ is the Sobolev space. The norms of spaces $(H^s(\Omega))^2$ and V^s will be denoted by $\|\cdot\|_s$. Denote by A_0 the operator defined by $A_0\mathbf{y} = P((\mathbf{f}_1\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{f}_2)$. Then operator $\mathcal{A} = \nu A + A_0 : D(\mathcal{A})(= V \cap (H^2(\Omega))^2) \to H$ can be defined on the space H with image in space $(D(\mathcal{A}^*))'$, which is the dual space of $D(\mathcal{A}^*)(= D(\mathcal{A}))$, by transition

$$\langle\mathcal{A}\mathbf{y},\mathbf{z}\rangle_{((D(\mathcal{A}^*))',D(\mathcal{A}^*))} := \langle\mathbf{y},\mathcal{A}^*\mathbf{z}\rangle.$$

Here, $\langle\cdot,\cdot\rangle_{((D(\mathcal{A}^*))',D(\mathcal{A}^*))}$ denotes the dual pair of the space $(D(\mathcal{A}^*))'$ and $D(\mathcal{A}^*)$. We still denote by $\mathcal{A}$ the extended operator. The operator $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ which is defined by

$$\mathcal{A}^*\mathbf{z} := \nu A\mathbf{z} + P(-(\mathbf{f}_1\cdot\nabla)\mathbf{z} + \mathbf{z}\cdot(\nabla\mathbf{f}_2)^T), \forall\mathbf{z} \in D(\mathcal{A}).$$

Define the operator $D : (H^s(\partial\Omega))^2 \to (H^{\frac{1}{2}+s}(\Omega))^2, s \geq 1/2$, by the solution $\mathbf{z} = D\mathbf{u}$ to the following equation

$$\begin{cases} k\mathbf{z} - \nu\triangle\mathbf{z} + (\mathbf{f_1}\cdot\nabla)\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{f_2} + \nabla p_1 = 0 & \text{in } \Omega, \\ \text{div}\mathbf{z} = 0 & \text{in } \Omega, \\ \mathbf{z} = \mathbf{u} & \text{on } \partial\Omega. \end{cases} \tag{3.1.2}$$

We see that $\mathbf{y}$ is the solution to the following equation (see (16), pp.114-115),

$$\frac{d\mathbf{y}(t)}{dt} + \mathcal{A}\mathbf{y}(t) = \mathcal{A}_k D\mathbf{u}(t) + \mathbf{f}(t), \qquad t \in (0,T); \qquad \mathbf{y}(0) = \mathbf{y}(T), \tag{3.1.3}$$

where $\mathcal{A}_k = kI + \mathcal{A}$. Equivalently,

$$\frac{d}{dt}\langle \mathbf{y}(t), \mathbf{z}\rangle + \langle y(t), \mathcal{A}^*\mathbf{z}\rangle = \langle \mathbf{u}(t), D^*\mathcal{A}_k^*\mathbf{z}\rangle_U + \langle \mathbf{f}(t), \mathbf{z}\rangle, \forall \mathbf{z} \in D(\mathcal{A}), \forall t \in (0,T). \tag{3.1.4}$$

Here, $\langle \cdot, \cdot \rangle_U$ denotes the inner product in the control space U, which is defined as the same as that in space $(L^2(\partial\Omega))^2$. It is easy to see, via Green's formula, that the dual $D^*\mathcal{A}_k^*$ of the operator $\mathcal{A}_k D$ is given by (see (16), Lemma 3.3.1)

$$D^*\mathcal{A}_k^*\mathbf{z} = -\nu\frac{\partial \mathbf{z}}{\partial \mathbf{n}}, \forall \mathbf{z} \in D(\mathcal{A}). \tag{3.1.5}$$

3.2 Existence of Optimal Solutions

A pair $(\mathbf{y}, \mathbf{u}) \in L^2(0,T; V^1) \times L^2(0,T; U)$ is called the optimal solution to optimal control problem (P) if it is the solution to the equation (3.1.3) and minimizes the cost functional $J(\cdot, \cdot)$.

If 1 is an eigenvalue of the operator $e^{-\mathcal{A}T}$, we denote by $\{\Psi_i\}_{i=1}^N, \{\Psi_i^*\}_{i=1}^N$ the (normalized) linearly independent eigenfunctions corresponding to eigenvalue 1 of $e^{-\mathcal{A}T}$ and $e^{-\mathcal{A}^*T}$, respectively. To get the existence of admissible pairs, we need the following assumptions for operator $e^{-\mathcal{A}T}$

(H1): The finite-dimensional spectral assumption (FDSA): We assume that, for the eigenvalue 1 of operator $e^{-\mathcal{A}T}$, algebraic and geometric multiplicity coincide.

(H2): The solution to the equation

$$\begin{cases} \frac{d\mathbf{z}(t)}{dt} - \mathcal{A}^*\mathbf{z}(t) = 0, & t \in (0,T), \\ \mathbf{z}(T) = \mathbf{z}(0), \\ \mathbf{z}|_{\partial\Omega} = 0, \end{cases} \tag{3.2.6}$$

is 0 when $\frac{\partial \mathbf{z}}{\partial \mathbf{n}} = 0$ on $(0,T) \times \partial\Omega$.

Now, we can prove the existence of optimal solutions.

Theorem 3.2.1. *There exists at least one optimal solution for optimal control problem (P) under assumptions (H1) and (H2).*

3.3 Maximum Principle for Optimality

Define the operator $\tilde{D} : L^2(0,T;U) \to L^2(0,T;H)$ by the solution $\tilde{\mathbf{z}} = \tilde{D}\mathbf{u}$ to the following equation

$$\frac{d\tilde{\mathbf{z}}}{dt} + \mathcal{A}_k \tilde{\mathbf{z}}(t) = \mathcal{A}_k D\mathbf{u}(t), \quad t \in (0,T); \quad \tilde{\mathbf{z}}(0) = \tilde{\mathbf{z}}(T). \tag{3.3.7}$$

When $\mathbf{u} \in \mathscr{U} (:= W^{1,2}([0,T];(H^{1/2}(\partial\Omega))^2))$, we can rewrite the equation (3.3.7) as

$$\frac{d\mathbf{z}}{dt} + \mathcal{A}_k \mathbf{z} = -\frac{d(D\mathbf{u})}{dt}, \qquad t \in (0,T); \qquad \mathbf{z}(0) = \mathbf{z}(T), \tag{3.3.8}$$

where $\mathbf{z} = \tilde{\mathbf{z}} - D\mathbf{u}$. Since the operator $\mathcal{A}_k$ is dissipative and generates a compact semigroup, it follows by Schauder fixed point Theorem that equation (3.3.8) admits a unique periodic solution $\mathbf{z} \in L^2(0,T;V) \cap C([0,T];H)$, which yields that equation (3.3.7) has a unique periodic solution $\tilde{\mathbf{z}} \in L^2(0,T;V^1)$ for each $\mathbf{u} \in \mathscr{U}$, and

$$\|\tilde{\mathbf{z}}\|_{L^2(0,T;V^1)} \le C\|\mathbf{u}\|_{\mathscr{U}}.$$

Here, $\|\cdot\|_{\mathscr{U}}$ denotes the norm of the Sobolev space $\mathscr{U}$. Therefore, the operator $\tilde{D}$ is continuous from $\mathscr{U}$ to $L^2(0,T;V^1)$, and so the adjoint operator $\tilde{D}^*$ is continuous from $L^2(0,T;(V^1)')$ to $\mathscr{U}'$, where $\mathscr{U}'$ is the dual space of $\mathscr{U}$ with pivot space $L^2(0,T;U)$.

Define the operator $\tilde{\triangle} : V^1 \to (V^1)'$ by

$$\langle \tilde{\triangle}\mathbf{y}, \mathbf{z}\rangle_{((V^1)',V^1)} := \langle \nabla\mathbf{y}, \nabla\mathbf{z}\rangle.$$

It is easy to check that the operator $\tilde{\triangle}$ is linear continuous from V^1 to $(V^1)'$.

For the necessary condition of the optimal pair $(\mathbf{y}^*, \mathbf{u}^*)$, we have the following Theorem.

Theorem 3.3.1. *Let $(\mathbf{y}^*, \mathbf{u}^*)$ be the optimal pair of the optimal control problem (P); then there exists $\mathbf{q} \in L^2(0,T;(V^1)')$ such that*

$$\begin{cases} \frac{d\mathbf{q}}{dt}(t) - \mathcal{A}^*\mathbf{q}(t) = \tilde{\triangle}(\mathbf{y}^*(t) - \mathbf{y}^0(t)), & \text{in } (0,T), \\ \mathbf{q}(0) = \mathbf{q}(T), & \\ \tilde{D}^*\mathbf{q}(t) = \frac{1}{k}(\tilde{D}^*\tilde{\triangle}(\mathbf{y}^*(t) - \mathbf{y}^0(t)) - \mathbf{u}^*(t)), & a.e. \text{ in } (0,T). \end{cases} \tag{3.3.9}$$

More precisely, we have $\mathbf{q}(t) = \mathbf{q}_1(t) + \mathbf{q}_2(t)$, where $\mathbf{q}_1 \in L^2(0,T;V) \cap W^{1,2}([0,T];V')$, $\mathbf{q}_2 \in L^2(0,T;(V^1)')$ and $\mathbf{q}_1, \mathbf{q}_2$ satisfy the following equations

$$\begin{cases} \frac{d\mathbf{q}_1}{dt}(t) - \mathcal{A}^*\mathbf{q}_1(t) = \tilde{\triangle}(\mathbf{y}^*(t) - \mathbf{y}^0(t)), & \text{in } (0,T), \\ \frac{d\mathbf{q}_2}{dt}(t) - \mathcal{A}^*\mathbf{q}_2(t) = 0, & \text{in } (0,T), \\ \tilde{D}^*(\mathbf{q}_1(t) + \mathbf{q}_2(t)) = \frac{1}{k}(\tilde{D}^*\tilde{\triangle}(\mathbf{y}^*(t) - \mathbf{y}^0(t)) - \mathbf{u}^*(t)), & a.e. \text{ in } (0,T). \end{cases} \tag{3.3.10}$$

3.4 Optimal Control for a Plane-Periodic Flow in 2-D Channel

Consider a laminar flow in a two dimensional channel with the walls located at $y = 0, 1$. We shall assume that the velocity field $\mathbf{z}(t,x,y) = (g(t,x,y), h(t,x,y))$ and the pressure $p(t,x,y)$ are 2π periodic in $x \in (-\infty, +\infty)$.

The dynamic of the flow is governed by the incompressible 2-D Stokes-Oseen Equations

$$\begin{cases} g_t - \nu\triangle g + g_x U + hU' = p_x + f_1, & y \in (0,1), x, t \in \mathbb{R}, \\ h_t - \nu\triangle h + h_x U = p_y + f_2, & y \in (0,1), x, t \in \mathbb{R}, \\ g_x + h_y = 0, & y \in (0,1), x, t \in \mathbb{R}, \\ g(t, x+2\pi, y) \equiv g(t,x,y), h(t,x+2\pi,y) \equiv h(t,x,y), & y \in (0,1), x, t \in \mathbb{R}, \\ g(t+T, x, y) \equiv g(t,x,y), h(t+T,x,y) \equiv h(t,x,y), & y \in (0,1), x, t \in \mathbb{R}. \end{cases} \tag{3.4.11}$$

Here, we consider a steady-state flow $\mathbf{U}(x,y)$ as a solution to the Navier-Stokes equation with zero vertical velocity component, i.e. $\mathbf{U}(x,y) = (U(x,y), 0)$. Since the flow is freely divergent, we have $U_x \equiv 0$, and so $U(x,y) \equiv U(y)$. This yields that

$$U(y) = C(y^2 - y), \forall y \in (0,1),$$

where $C \in \mathbb{R}^-$. In the following, we take $C = -\frac{a}{2\nu}$, where $a \in \mathbb{R}^+$. (See(17), Section 3.5; see also (18))

Here, we want to apply Theorem 3.3.1 to obtain the necessary conditions for the optimal pair of optimal control problem (P) governed by system (3.4.11). To this aim, we recall first the Fourier functional setting for description of periodic fluid flows in the channel $(-\infty, +\infty) \times (0,1)$.

Let $L^2_\pi(Q), Q = (0, 2\pi) \times (0,1)$ be the space of all functions $g \in L^2_{loc}(\mathbb{R} \times (0,1))$ which are $2\pi-$periodic in x. Similarly, $H^1_\pi(Q), H^2_\pi(Q)$ are defined. For instance,

$$H^1_\pi(Q) := \left\{ g; g = \sum_k a_k(y) e^{ikx}, a_k = \bar{a}_{-k}, a_0 = 0, \sum_k \int_0^1 (k^2|a_k|^2 + |a_k'|^2) dy < \infty \right\}.$$

We set

$$H := \left\{ (g,h) \in (L^2_\pi(Q))^2; g_x + h_y = 0, h(x,0) = h(x,1) = 0 \right\}.$$

If $g_x + h_y = 0$, then, the trace of (g,h) at $y = 0, 1$ is well defined as an element of $H^{-1}(0,2\pi) \times H^{-1}(0,2\pi)$(see, e.g., (12)).

We also set

$$V := \left\{ (g,h) \in H \cap (H^1_\pi(Q))^2; g(x,0) = g(x,1) = h(x,0) = h(x,1) = 0 \right\}.$$

As defined above, the space $L^2_\pi(Q)$ is, in fact, the factor space $L^2_\pi(Q)/Z$.

Let $P : (L^2_\pi(Q))^2 \to H$ be the Leray projector and $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be the operator defined by

$$\mathcal{A}(g,h) := P(-\nu\triangle g + g_x U + hU', -\nu\triangle h + h_x U), \forall (g,h) \in D(\mathcal{A}) = H^2((0,2\pi) \times (0,1)). \tag{3.4.12}$$

We associate with (3.4.11) the boundary value conditions

$$\begin{aligned} g(t,x,0) = u^0(t,x), g(t,x,1) = u^1(t,x), t \geq 0, x \in \mathbb{R}, \\ h(t,x,0) = v^0(t,x), h(t,x,1) = v^1(t,x), t \geq 0, x \in \mathbb{R}, \end{aligned} \tag{3.4.13}$$

and, for $k > 0$ sufficiently large, we can define the Dirichlet map $D : X \to H$ by $D(u, v) := (\tilde{g}, \tilde{h})$,

$$\begin{cases} k\tilde{g} - \nu\triangle\tilde{g} + \tilde{g}_x U + hU' = p_x, & y \in (0,1), x \in \mathbb{R}, \\ k\tilde{h} - \nu\triangle\tilde{h} + \tilde{h}_x U = p_y, & y \in (0,1), x \in \mathbb{R}, \\ \tilde{g}_x + \tilde{h}_y = 0, & y \in (0,1), x \in \mathbb{R}, \\ \tilde{g}(x+2\pi, y) \equiv \tilde{g}(x,y), \tilde{h}(x+2\pi, y) \equiv \tilde{h}(x,y), & y \in (0,1), x \in \mathbb{R}, \\ \tilde{g}(x,y) = u(x,y), \tilde{h}(x,y) = v(x,y), & y = 0, 1, \ x \in \mathbb{R}. \end{cases} \tag{3.4.14}$$

Here,

$$X = \{(u,v) \in L^2((0,2\pi) \times \partial(0,1)); u(x+2\pi, y) = u(x,y),$$
$$v(x+2\pi, y) = v(x,y), v(x,0) = v(x,1) = 0, \forall x \in (0,2\pi)\}.$$

Then system (3.4.11) with boundary conditions (3.4.13) can be written as

$$\frac{d}{dt}\mathbf{z}(t) + \mathcal{A}\mathbf{z}(t) = \mathcal{A}_k D\mathbf{u}(t) + \mathbf{f}(t), t \geq 0; \qquad \mathbf{z}(0) = \mathbf{z}(T), \tag{3.4.15}$$

where $\mathbf{z} = (g,h), \mathbf{u} = (u,v), \mathbf{f} = (f_1, f_2)$. We denote again by $\mathcal{A}$ the extension of $\mathcal{A}$ on the complexified space $\tilde{H}$ and denote by $\mathcal{A}^*$ the dual operator of $\mathcal{A}$. In order to apply Theorem 3.3.1, we shall show in following lemma that assumption (H2) holds true in this case.

Lemma 3.4.1. *The solution to the equation*

$$\begin{cases} \frac{d\mathbf{z}(t)}{dt} - \mathcal{A}^*\mathbf{z}(t) = 0, & t \in (0,T), \\ \mathbf{z}(T) = \mathbf{z}(0), \\ \mathbf{z}|_{\partial Q} = 0, \end{cases} \tag{3.4.16}$$

is 0 when $\frac{\partial \mathbf{z}(x,y)}{\partial \mathbf{n}} = 0$ *for* $x \in (0,2\pi), y = 0, 1,$ *where* $\mathcal{A}$ *is defined as in (3.4.12).*

If $1 \in \sigma(e^{-\mathcal{A}T})$, and Φ is the corresponding eigenfunction, then $\Phi = \mathbf{w}(0)$, where $\mathbf{w}(t,x,y) = (g(t,x,y), h(t,x,y))$ is the solution to the equation

$$\begin{cases} g_t - \nu\triangle g + g_x U + hU' = p_x, & y \in (0,1), x, t \in \mathbb{R}, \\ h_t - \nu\triangle h + h_x U = p_y, & y \in (0,1), x, t \in \mathbb{R}, \\ g_x + h_y = 0, & y \in (0,1), x, t \in \mathbb{R}, \\ g(t, x+2\pi, y) \equiv g(t,x,y), h(t, x+2\pi, y) \equiv h(t,x,y), & y \in (0,1), x, t \in \mathbb{R}, \\ g(t+T, x, y) \equiv g(t,x,y), h(t+T, x, y) \equiv h(t,x,y), & y \in (0,1), x, t \in \mathbb{R}. \end{cases} \tag{3.4.17}$$

Using the same arguments applied in the proof of Lemma 3.4.1, we can reduce equation (3.4.17) to be

$$-\nu h_{kl}^{iv} + (2\nu k^2 + ikU + \frac{2j\pi l}{T})h_{kl}^{''} - k(\frac{2j\pi lk}{T} + \nu k^3 + ik^2 + iU^{''})h_{kl} = 0, y \in (0,1). \tag{3.4.18}$$

Denote by Σ the subspace of H which is defined by

$$\Sigma := \{(\textstyle\sum_{k,l} \frac{i}{k} e^{ikx} h'_{kl}(y), \sum_{k,l} e^{ikx} h_{kl}(y)); h_{kl}(y) \text{ solution to equation (3.4.18)}\}.$$

In fact, Σ is the space spanned by the eigenfunctions corresponding to eigenvalue 1. We shall assume, in this case, the following assumption holds.

(H1') dim(Σ) equal to the algebraic multiplicity of eigenvalue 1.

With this assumption and the result we obtained in Lemma 3.4.1, we have the following result.

Theorem 3.4.1. *There exists optimal solution* $(\mathbf{z}^*, \mathbf{u}^*)$ *to the optimal control problem* (P) *governed by system (3.4.11) (or 3.4.15), and there exists* $\mathbf{q} \in L^2(0,T;(V^1)')$ *such that* $\mathbf{q}, \mathbf{z}^*, \mathbf{u}^*$ *satisfy system (3.3.9).*

4

Boundary optimal feedback controller for time-periodic Stokes-Oseen equations

4.1 Introduction

Here we shall consider the optimal control problem :

$$(P_0) \quad Min\{\frac{1}{2}\int_0^T\int_\Omega |\mathbf{y}|^2 d\mathbf{x}dt + \frac{1}{2}\int_0^T\int_\Gamma |\mathbf{u}|^2 d\mathbf{x}dt\}$$

subject to the periodic Stokes-Oseen equations with Dirichlet boundary condition:

$$\begin{cases} \frac{d\mathbf{y}}{dt} - \nu\triangle\mathbf{y} + (\mathbf{f}_1\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{f}_2 + \nabla p = \mathbf{f}_0 & \text{in } \Omega\times(0,T), \\ \mathbf{y}(0) = \mathbf{y}(T) & \text{in } \Omega, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } \Omega\times(0,T), \\ \mathbf{y} = \mathbf{u} & \text{on } \Gamma\times(0,T). \end{cases} \tag{4.1.1}$$

Here Ω is a bounded open subset with smooth boundary Γ in $\mathbb{R}^2$. Function $\mathbf{u}(t)\in U, \forall t\in(0,T)$ is the boundary control, where $U = V^0(\Gamma) \doteq \{\mathbf{u}\in(L^2(\Gamma))^2; \langle\mathbf{u}\cdot\mathbf{n},1\rangle_{((H^{-1/2}(\partial\Omega))^2,(H^{1/2}(\partial\Omega))^2)} = 0\}$. Here $\mathbf{n}$ is the outer normal vector of Γ, $\mathbf{f}_i(\mathbf{x}), i = 1,2$ are steady state functions and $\mathbf{f}_i\in(W^{2,\infty}(\Omega))^2\cap V^0(\Omega)$, where space $V^0(\Omega)$ is defined by

$$V^0(\Omega) = \{\mathbf{y}\in(L^2(\Omega))^2; \nabla\cdot\mathbf{y} = 0, \langle\mathbf{y}\cdot\mathbf{n},1\rangle_{((H^{-1/2}(\Gamma))^2,(H^{1/2}(\Gamma))^2)} = 0\}.$$

The space $V^0(\Omega)$ is a closed subspace of $(L^2(\Omega))^2$, and it is a Hilbert space with the scalar product

$$\langle\mathbf{y},\mathbf{z}\rangle = \int_\Omega \mathbf{y}\cdot\mathbf{z}d\mathbf{x},$$

and the corresponding norm $|\mathbf{y}| = (\int_\Omega|\mathbf{y}|^2 d\mathbf{x})^{1/2}$.(We shall denote by the same symbol $|\cdot|$ the norms in $\mathbb{R}^2$, $L^2(\Omega)^2$ and $V^0(\Omega)$. The scalar product in $(L^2(\Omega))^2$ is the same as in $V^0(\Omega)$, which will be also denoted by $\langle\cdot,\cdot\rangle$ if there is no ambiguous.) Define the spaces

$$V^s(\Omega) = (H^s(\Omega))^2\cap V^0(\Omega), s\geq 0,$$

$$V^s_n(\Omega) = \{\mathbf{y} \in (H^s(\Omega))^2, \nabla \cdot \mathbf{y} = 0, \mathbf{y} \cdot \mathbf{n} = 0\}, s \geq 0,$$
$$V^s_0(\Omega) = \{\mathbf{y} \in (H^s(\Omega))^2, \nabla \cdot \mathbf{y} = 0, y = 0 \text{ on } \Gamma\}, s \geq 1/2,$$
$$V^s(\Gamma) = \{\mathbf{u} \in (H^s(\Gamma))^2, \langle \mathbf{u} \cdot \mathbf{n}, 1\rangle_{((H^{-1/2}(\Gamma))^2,(H^{1/2}(\Gamma))^2)} = 0\}, s \geq -1/2,$$
$$V^s_n(\Gamma) = \{\mathbf{u} \in V^s(\Gamma), \mathbf{u} \cdot \mathbf{n} = 0\}, s \geq -1/2,$$

where $H^s(\Omega), H^s(\Gamma)$ are the Sobolev spaces. The norms of spaces $(H^s(\Omega))^2$ and V^s will be denoted by $\|\cdot\|_s$. The norm $\|\cdot\|_1$ will denote by $\|\cdot\|$ for simplicity. We shall also denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the norm and inner product respectively of spaces $L^2(\Gamma), V^0(\Gamma)$ and $V^0_n(\Gamma)$. We shall use the following notation $Q_T = \Omega \times (0,T), \Sigma_T = \Gamma \times (0,T)$. For spaces of time dependent functions we set

$$V^{s,\sigma}(Q_T) = H^\sigma(0,T; V^0(\Omega)) \cap L^2(0,T; V^s(\Omega)),$$
$$V^{s,\sigma}(\Sigma_T) = H^\sigma(0,T; V^0(\Sigma)) \cap L^2(0,T; V^s(\Sigma)).$$

For all $\psi \in H^{1/2+\varepsilon}(\Omega)$ with $\varepsilon > 0$, we denote by $c(\psi)$ the constants defined by

$$c(\psi) = \frac{1}{|\Gamma|}\int_\Gamma \psi. \tag{4.1.2}$$

Let us define the operators $\gamma_\tau \in \mathcal{L}(V^0(\Gamma)), \gamma_n \in \mathcal{L}(V^0(\Gamma))$ by

$$\gamma_\tau \mathbf{u} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \gamma_n u = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = \mathbf{u} - \gamma_\tau \mathbf{u}.$$

Lemma 4.1.1. *The operators γ_τ and γ_n satisfy*

$$\gamma_\tau = \gamma_\tau^*,\ \gamma_n = \gamma_n^*,\ \text{and } (I-P)D = (I-P)D\gamma_n.$$

Lemma 4.1.2. *The operator*

$$R_A = D^*(I-P)D + I$$

is an isomorphism from $V^0(\Gamma)$ into itself. Moreover, for all $0 \leq s \leq 3/2$, its restriction to $V^s(\Gamma)$ is an isomorphism from $V^s(\Gamma)$ into itself. In addition R_A satisfies

$$R_A\gamma_n = \gamma_n R_A \gamma_n, R_A\gamma_\tau = \gamma_\tau R_A \gamma_\tau = \gamma_\tau.$$

The restriction of R_A to $V^0_\tau(\Gamma)$ is an isomorphism from $V^0_\tau(\Gamma)$ into itself, and we have

$$R_A^{-1}\mathbf{u} = (\gamma_n R_A \gamma_n)^{-1}\mathbf{u} = \gamma_n R_A^{-1}\mathbf{u}, \forall \mathbf{u} \in V^0_\tau(\Gamma).$$

We introduce the operators $F_n = \mathcal{A}_k PD\gamma_n, F_\tau = \mathcal{A}_k PD\gamma_\tau, F = F_n + F_\tau$. where $\mathcal{A}_k = kI + \mathcal{A}$.

Proposition 4.1.1. *For all $\mathbf{\Phi} \in D(A), B^*\mathbf{\Phi} \in V^{1/2}(\Gamma)$, we have*

$$F^*\mathbf{\Phi} = D^*\mathcal{A}_k^*\mathbf{\Phi}, F_\tau^*\mathbf{\Phi} = \gamma_\tau D^*\mathcal{A}_k^*\mathbf{\Phi}, F_n^*\mathbf{\Phi} = \gamma_n D^*\mathcal{A}_k^*\mathbf{\Phi},$$

and

$$F^*\mathbf{\Phi} = -\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n}, F_\tau^*\mathbf{\Phi} = -\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}, F_n^*\Phi = \psi\mathbf{n} - c(\psi)\mathbf{n},$$

with

$$\nabla\psi = (I-P)(\nu\triangle\mathbf{\Phi} + (\mathbf{f_1}\cdot\nabla)\mathbf{\Phi} + (\nabla\mathbf{f_2})^T\mathbf{\Phi}).$$

We rewrite equation (4.1.1) in the form

$$P\mathbf{y}' + \mathcal{A}P\mathbf{y} = F\mathbf{u} + \mathbf{f}, \mathbf{y}(0) = \mathbf{y}(T), \tag{4.1.3}$$

$$(I-P)\mathbf{y} = (I-P)D\gamma_n\mathbf{u}. \tag{4.1.4}$$

where $\mathbf{f} = P\mathbf{f}_0$.

From the argument in (19), we can rewrite the cost functional as following

$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^T \int_\Omega |P\mathbf{y}|^2 d\mathbf{x}dt + \frac{1}{2}\int_0^T \int_\Gamma (|R_A^{1/2}\gamma_n\mathbf{u}|^2 + |\gamma_\tau\mathbf{u}|^2) d\mathbf{x}dt,$$

The control problem (P_0) is equivalent to

(P) $\qquad \inf\{J(\mathbf{y},\mathbf{u})|(\mathbf{y},\mathbf{u}) \text{ satisfies } (4.1.3),\ \mathbf{u} \in V^{0,0}(\Sigma_T)\}$

4.2 Existence of Optimal Solution and Maximum Principle

By optimal solution to optimal control problem(OCP) (P) we mean a pair $(\mathbf{y},\mathbf{u}) \in L^2(0,T;V^0(\Omega)) \times L^2(0,T;V^0(\Gamma))$ solution to (4.1.3) and minimize the cost functional $J(\mathbf{y},\mathbf{u})$.

As stated in Section 4.3, the existence of periodic solution to equation (4.1.1) does not hold in general, and even the existence of admissible pair for the optimal control problem is not trival.

Theorem 4.2.1. *There exists a unique optimal solution $(\mathbf{y}^*,\mathbf{u}^*)$ for optimal control problem (P) under assumptions (H1) and (H2).*

We can also derive the maximum principle for optimal solution $(\mathbf{y}^*,\mathbf{u}^*)$.

Theorem 4.2.2. *The admissible pair $(\mathbf{y}^*,\mathbf{u}^*)$ solution to OCP (P) if and only if there is function $\mathbf{q} \in L^2(0,T;D(A)) \cap H^1([0,T],V_n^0(\Omega))$ satisfies together with $(\mathbf{y}^*,\mathbf{u}^*)$ the following system*

$$\begin{cases} \frac{d\mathbf{q}}{dt}(t) - \mathcal{A}^*\mathbf{q}(t) = P\mathbf{y}^*, & t \in (0,T), \\ \mathbf{q}(0) = \mathbf{q}(T), & \\ \mathbf{u}^*(t) = F_\tau^*\mathbf{q}(t) + R_A^{-1}F_n^*\mathbf{q}(t), & a.e. t \in (0,T). \end{cases} \tag{4.2.1}$$

More precisely, we have $\mathbf{q}(t) = \mathbf{q}_1(t) + \mathbf{q}_2(t)$, where $\mathbf{q}_1,\mathbf{q}_2 \in V^{2,1}(Q_T)$, and $\mathbf{q}_1,\mathbf{q}_2$ satisfy the following equation

$$\begin{cases} \frac{d\mathbf{q}_1}{dt}(t) - \mathcal{A}^*\mathbf{q}_1(t) = P\mathbf{y}^*(t), & t \in (0,T), \\ \frac{d\mathbf{q}_2}{dt}(t) - \mathcal{A}^*\mathbf{q}_2(t) = 0, & t \in (0,T), \\ \mathbf{u}^*(t) = F_\tau^*\mathbf{q}(t) + R_A^{-1}F_n^*\mathbf{q}(t), & a.e. t \in (0,T). \end{cases} \tag{4.2.2}$$

Moreover, we can obtain the following regularity properties for the optimal solution via the Euler-Lagrange system.

Theorem 4.2.3. *Let $\mathbf{f} \in L^2(0,T;(V^{2\varepsilon}(\Omega))')$, then the optimal solution $(\mathbf{y}^*,\mathbf{u}^*)$ satisfy $\mathbf{u}^* \in V^{1-\varepsilon,1/2-\varepsilon}(\Sigma_T)$, $P\mathbf{y}^* \in V^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T)$, and the following estimate holds*

$$\|P\mathbf{y}^*\|_{V^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T)} \le C\|\mathbf{f}\|_{L^2(0,T;(V^{2\varepsilon}(\Omega))')}. \tag{4.2.3}$$

4.3 Feedback control and application

In the sequel we shall only consider the tangential Dirichlet control for the convenient to present the idea of the feedback synthesis and it's application. Denote here by H the space $V_n^0(\Omega)$.

To get the optimal feedback controller of optimal control problem (P), we consider the dual optimal control problem:

$$\begin{array}{c}\min\{\int_0^T \left(\frac{1}{2}(|F^*\mathbf{p}(t)|^2 + |\mathbf{y}(t)|^2) + \langle \mathbf{f}(t), \mathbf{p}(t)\rangle\right) dt; \\ \mathbf{p}' - \mathcal{A}^*\mathbf{p} = y, \mathbf{p}(0) = \mathbf{p}(T), \mathbf{y} \in L^2(0,T;H)\}.\end{array} \tag{4.3.1}$$

Equivalently,

$$\begin{array}{c}\min\{\int_0^T \left(\frac{1}{2}(|F^*\mathbf{p}(t)|^2 + |\mathbf{y}(t)|^2) + \langle \mathbf{f}(T-t), \mathbf{p}(t)\rangle\right) dt; \\ \mathbf{p}' + \mathcal{A}^*\mathbf{p} = \mathbf{y}, \mathbf{p}(0) = \mathbf{p}(T), \mathbf{y} \in L^2(0,T;H)\}.\end{array} \tag{4.3.2}$$

If $(\mathbf{p}, \mathbf{y})$ is an optimal pair in problem (4.3.1), then it follows via maximum principle and duality arguments that $(-\mathbf{y}(T-t), -F^*\mathbf{p}(T-t))$ is optimal in problem (P).

The dynamic programming equation corresponding to problem (4.3.2) is

$$\psi_t(t,\mathbf{p}) + \frac{1}{2}|\psi_{\mathbf{p}}(t,\mathbf{p})|^2 + \langle \mathcal{A}^*\mathbf{p}, \psi_{\mathbf{p}}(t,\mathbf{p})\rangle = \frac{1}{2}|F^*\mathbf{p}|^2 + \langle \mathbf{f}(t), \mathbf{p}\rangle, \tag{4.3.3}$$

where $\psi_{\mathbf{p}} = \nabla_p \psi$.

For the existence of solution to system (4.3.3), we have the following Theorem.

Theorem 4.3.1. *There is a continuous function $\psi : [0,T] \times H \to \mathbb{R}$ which is convex and Gateaux differentiable in $\mathbf{p}$, absolutely continuous in t for each $\mathbf{p} \in D(A)$, satisfies a.e. in $(0,T)$ Eq. (4.3.2), and*

$$\psi(0,\mathbf{p}) - \psi(T,\mathbf{p}) = \mu, \tag{4.3.4}$$

for some $\mu \in \mathbb{R}$.

Before proving Theorem 4.3.1, we pause briefly to present a few consequences.

Consider the feedback law,

$$\mathbf{z}(t) = -\psi_{\mathbf{p}}(T-t, \mathbf{p}(t)),\ t \in (0,T), \tag{4.3.5}$$

where $\mathbf{p}$ is the solution to closed loop system

$$\mathbf{p}' - \mathcal{A}^*\mathbf{p} + \psi_{\mathbf{p}}(T-t, \mathbf{p}(t)) = 0, t \in (0,T); \mathbf{p}(0) = \mathbf{p}(T). \tag{4.3.6}$$

We shall assume here that equation 4.3.6 has at least one solution (see remarks of Theorem 2 in (20). If $\mathbf{p}^*$ is a solution to equation (4.3.6), then by a standard calculation involving equation (4.3.4) it follows that the pair $(\mathbf{p}^*(t), -\psi_{\mathbf{p}}(T-t, \mathbf{p}^*(t)))$ is the optimal in the dual problem (4.3.2) and so as mentioned earlier the pair

$$\mathbf{y}^*(t) = \psi_{\mathbf{p}}(t, \mathbf{p}^*(T-t)), \mathbf{u}^*(t) = -F^*\mathbf{p}^*(T-t) \tag{4.3.7}$$

is optimal in problem (P). In other words,

$$\mathbf{u}^*(t) = -F^*\psi^*_{\mathbf{y}}(t, \mathbf{y}^*(t)), t \in (0,T) \tag{4.3.8}$$

is an optimal feedback controller for problem (P). Here ψ^* is the conjugate of the function $\mathbf{p} \to \psi(t, \mathbf{p})$. Summarizing we have

Theorem 4.3.2. *Assume that the periodic problem (4.3.6) has a mild solution $\mathbf{p} \in C([0,T]; H)$. Then the feedback control (4.3.8) is optimal in problem (P).*

In the following, we give an application of the optimal synthesis results obtained above.

Consider now the periodic Navier-Stokes equation with homogeneous Dirichlet boundary condition

$$\begin{cases} -\nu\triangle\mathbf{y}_e + (\mathbf{y}_e\cdot\nabla)\mathbf{y}_e + \nabla p = \mathbf{f}_0 & \text{in } \Omega, \\ \nabla\cdot\mathbf{y}_e = 0 & \text{in } \Omega, \\ \mathbf{y}_e = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.9}$$

We assume that the solution $\mathbf{y}_e$ is regular enough. When the force term $\mathbf{f}_0$ is perturbed by another time-periodic force $\mathbf{f}$, the perturbed periodic solution satisfies the following equation

$$\begin{cases} \mathbf{y}'(t) - \nu\triangle\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f}_0 + \mathbf{f}(t) & \text{in } \Omega, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } \Omega, \\ \mathbf{y}(0) = \mathbf{y}(T) & \text{in } \Omega, \\ \mathbf{y} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.10}$$

However, this perturbed periodic solution may do not stay near the original solution no matter how small the perturbation is. It is kind of property of non-continuity dependent of the outer force term for the periodic Navier-Stokes equation. We shall prove below that we can put a boundary feedback control to overcome this defects for each perturbation $\mathbf{f}$ small enough. Still denote $\mathcal{A}\mathbf{y} = A\mathbf{y} + P((\mathbf{y}_e\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y}_e)$. We have the following satbilization result.

Theorem 4.3.3. *For all $0 < \varepsilon < 1/4$, There is a constant $\mu_0 > 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that for each $\mu \in (0,\mu_0)$ and $\|f\|_{L^2(0,T;V^0(\Omega))} \le \eta(\mu)$, there is a feedback boundary input $\mathbf{u}(t) = -F^*\psi^*_{\mathbf{y}}(t, \mathbf{y}(t))$ such that the periodic equation*

$$\begin{cases} \frac{d\mathbf{y}}{dt} + \mathcal{A}\mathbf{y} + B\mathbf{y} = F\mathbf{u} + \mathbf{f}, \\ \mathbf{y}(0) = \mathbf{y}(T). \end{cases} \tag{4.3.11}$$

admits a unique solution, in the set

$$D_\mu = \{\mathbf{y} \in V^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T); \|\mathbf{y}\|_{V^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T)} \le \mu, \mathbf{y}(0) = \mathbf{y}(T)\}.$$

and the following estimate holds

$$\|\mathbf{y}\|_{V^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T)} \le C\|\mathbf{f}\|_{L^2(0,T;V^0(\Omega))}. \tag{4.3.12}$$

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